On the Evaluation of Gluon Condensate Effects in the Holographic Approach to QCD

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Abstract

In holographic QCD the effects of gluonic condensate can be encoded in a suitable deformation of the 5D metric. We develop two different methods for the evaluation of first order perturbative corrections to masses and decay constants of vector resonances in 5D Hard-Wall models of QCD due to small deformations of the metric. They are extracted either from a novel compact form for the first order correction to the vector two-point function, or from perturbation theory for vector bound-state eigenfunctions: the equivalence of the two methods is shown. Our procedures are then applied to flat and to AdS 5D Hard-Wall models; we complement results of existing literature evaluating the corrections to vector decay constant and to two-pion-one-vector couplings: this is particularly relevant to satisfy the sum rules. We concentrate our attention on the effects for the Gasser-Leutwyler coefficients; we show that, as in the Chiral Quark model, the addition of the gluonic condensate improves the consistency, the understanding and the agreement with phenomenology of the holographic model.

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1 Introduction

The original Maldacena conjecture [1] on the AdS/CFT duality between $\mathcal{N} = 4$ SYM and string theory on $AdS_5 \times S^5$ has been extended to a conjectured holographic equivalence between four-dimensional (4D) strongly coupled QCD at large N_c and a 5D weakly interacting gauge theory coupled to gravity on a 5D space M_5 not necessarily (even asymptotically) AdS_5. New ingredients have to be introduced in order to comply with the effective low energy description of a confining theory such as QCD. One possibility is to modify the background geometry of the dual gravity theory, as in the Sakai-Sugimoto model [2], or, as in the phenomenological Hard-Wall (HW) models of [3] and [4], one can cut-off the AdS_5 at a finite size, producing confinement and an infinite spectrum of Kaluza-Klein resonances. The two cited HW models differ in the implementation of spontaneous chiral symmetry breaking (χSB), another fundamental ingredient of low energy hadron dynamics. In [3], χSB is induced by a 5D scalar field, having the right properties to be the dual of the $\bar{q}q$ operator of QCD, whose non-vanishing vacuum expectation value (VEV) is responsible for χSB. In the approach of [4], χSB follows from imposing appropriate infrared (IR) boundary conditions.

The Soft-Wall (SW) model, proposed in [5], is instead a 5D holographic model in which the AdS_5 has no cut-off, but confinement and an infinite spectrum of Kaluza-Klein resonances are due a non-trivial dilaton background. The phenomenological advantage of the SW model is the fact that, contrary to the what happens for the KK states of the HW models, the spectrum of resonances follows a linear Regge-trajectory, which leads to a better agreement with the corresponding QCD resonance spectrum in the intermediate energy regime.

In any of models above, one boldly conjectures the applicability of the following holographic recipe for the calculation of correlation functions of the dual 4D theory. For every quantum operator $\mathcal{O}(x)$ in QCD, there exists a corresponding 5D field $\phi(x, z)$, fixed by the boundary condition $\phi(x, 0) \equiv \phi_0(x)$ on the ultraviolet (UV) boundary of M_5. The generating functional of the correlation functions of the 4D theory, is equal to the 5D action evaluated *on-shell*:

$$\exp\left(W_4[\phi_0(x)]\right) \equiv \langle \exp(i \int d^4x\, \phi_0(x)\, \mathcal{O}(x))\rangle_{\mathrm{QCD}_4} = \exp\left(iS_{\mathrm{M}_5}[\phi_0(x)]\right). \tag{1.1}$$

Varying $S_{\mathrm{M}_5}[\phi_0(x)]$ with respect to the sources $\phi_0(x)$, and then setting them to zero, one gets the connected n-point Green's functions of the 4D theory.

Having a 5D holographic description of low energy QCD makes compelling the comparison with 4D low energy QCD models: VMD, large N_C, Chiral Quark Model, ... QCD at low energy is described by chiral perturbation theory and thus the question amounts to how good is the prediction for the Gasser-Leutwyler coefficients, L_i. It is fair to say that all these models, 4D and 5D, compare fairly well with the L_i's phenomenological values: a more specific question is if the presence of gluonic condensate improves the agreement with phenomenology. Actually this question has a very neat answer in the Chiral Quark Model [6, 7]: the agreement with phenomenology improves [7]. So we think that this is a very well motivated question for holographic QCD.

In the holographic dictionary, 4D condensates, *i.e.* non-vanishing VEV's of some composite operators of the 4D theory, are related to the behavior near the UV boundary of the dual 5D fields [8]. The effect of a gluonic condensate can be induced by a non-trivial dilaton field, or, equivalently, by a suitable deformation of the original AdS_5 metric, as we will discuss in detail.

We shall be concerned with the correction to the two-point functions in presence of small deformations of the 5D metric background. Here small means that we can treat them as perturbations and we shall calculate the first order effect on the two-point function of the theory. Although applicable also to the evaluation of two-point functions of other 4D operators, we shall consider only the case of conserved (flavor) currents, $\langle J_\nu(x) J_\mu(y) \rangle$. So, instead of generic 5D actions, we shall consider only the case of HW models with 5D vector gauge fields, as they are the fields dual to these currents.

After a a short review of the holographic prescription in Sec. 2, we compute in Sec.3 the first order correction to the vector two-point function and show how to cast them in a very compact form. With hindsight, we justify the result deriving it directly from the holographic prescription. We present also another approach to evaluate directly how the metric deformation affects masses and decay constants of the vector resonances, which appear as intermediate states in the bound-state expansion of the two-point function. We do it by developing perturbation theory for the resonance wave-functions. The resulting formulae differ from those of usual perturbation theory, since a deformation of the metric affects also the scalar product of wave-functions. The interest of this approach lies not only in the fact that it provides a different route to the evaluation of the perturbation of resonance masses and decay constants, which could be alternatively extracted from the behavior of the perturbed two-point around its singularities: in fact, since it directly provides the perturbed wave-functions, it allows, for instance, the evaluation of the effects of the metric deformation on other low energy coupling such as the two-pion-one-vector coupling. The formal equivalence of the two methods is shown in the Appendix.

In Sect.4, we consider first the case of an HW model with flat 5D metric. It is computationally simpler and we use it to illustrate our methods in the evaluation of the corrections to resonance masses and decay constants: these results are original. We then examine the case of HW model where the extra-dimension is an AdS_5 slice [3], [4]. We also consider some effects on the axial vector resonance, for which we adopt the description in [4]. Moreover, perturbation theory for the resonance wave-function allows us to evaluate also the corrections to the two-pion-one-vector couplings. We want to stress that our results represent a novelty, since no expressions for the corrections to the vector decay constants and to the two-pion-one-vector couplings have appeared in the literature even in papers explicitly devoted to this effect in the AdS_5 HW model. This was one of the motivations of our work and constitutes the core of our original analytic work.

In holographic models, the fact that all vector resonance wave functions are Kaluza-Klein excitations of a 5D gauge field has the consequence that they satisfy completeness relations from which sum rules can be obtained. We address the issue of the saturation of these sum rules by the first few resonances both in the unperturbed and in the perturbed metric case. This relevant discussion is possible only after our novel calculation of the corrections to the vector decay constants and to the two-pion-one-vector couplings.

In Sec. 5 we present numerical analyses and conclusions: we evaluate the effects of the gluon condensate on low energy parameters of QCD, in particular the Gasser-Leutwyler coefficients, the pion decay constant and $\rho-$meson parameters. Our discussion leads to a remarkable analogy between the AdS_5 HW model and the Chiral Quark model.

2 Two-point function of vector gauge fields in Hard-Wall models

We consider a 5D holographic Hard-Wall models with the extra-dimension z restricted to a finite interval $0 \leq z \leq z_0$ and unperturbed metric written as a conformal factor times the 5D flat metric as follows:

$$g_{MN}dx^M dx^N = w(z)^2 \left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right), \tag{2.1}$$

where $\eta_{\mu\nu} = \mathrm{Diag}\,(1,-1,-1,-1)$ and $\mu,\nu = (0,1,2,3)$, $M,N = (0,1,2,3,z)$. The conformal factor $w(z)$ is left unspecified; in the case of AdS_5 it would be $w_{AdS}(z) = 1/z$. We consider $z = 0$ as the UV-brane, where the bulk fields have to coincide with the external sources of the 4D theory and $z = z_0$ as the IR-brane where suitable boundary conditions are imposed. As we shall mainly consider vector gauge fields, we impose Neumann-type boundary conditions, which are common to all the HW models mentioned in the introduction, where, moreover, these b.c. force the absence of a massless resonance mode. In the case of SW models, where the fifth coordinate z is no more restricted to a finite interval, the IR boundary condition would be replaced by a normalization condition.

The quadratic part of the 5D action for the gauge fields is given by:

$$S_{M_5} = -\frac{1}{4g_5^2}\int d^4x \int_0^{z_0} dz\sqrt{g}\, g^{MN}g^{RS}F_{MR}F_{NS} \tag{2.2}$$

with linearized field strength $F_{MN} = \partial_M A_M - \partial_N A_M$ and g_5^2 is a 5D coupling constant, which in AdS_5 models is usually fixed to $g_5^2 = 12\pi^2/N_c$, where N_c is the number of colors of the 4D dual gauge theory, by matching the two-point function at large Euclidean momentum with the perturbative result of QCD parton loop. Otherwise, g_5^2 could be fixed by the requirement of obtaining the physical value of the pion decay constant f_π. Its actual value is irrelevant for the general discussion of this and the next Sections, so we shall momentarily suppress it. For the same reason, we have also suppressed any flavor index of the 5D fields. We shall restore the correct factors in Sec. 4.2, when we discuss the 5D AdS slice, which is the playground of the HW models [3, 4]. We shall need them in order to make some numerical analysis of the effects of condensates and compare with the existing literature.

It is convenient to work in $A_z = 0$ gauge. Then, the 5D gauge fields $A_\mu(x,z)$ holographically correspond to conserved vector currents $J_\mu(x)$ of the dual 4D theory, and the holographic formula will allow the calculation of the correlation function of two currents, in two steps. First, one solves the 5D equations of motion of the gauge field, requiring the solution to coincide on the UV boundary with the 4D source $A_\mu(x)$ of the vector current. This is done by means of the bulk-to-boundary propagator. Secondly, the 5D action is evaluated on this solution and identified with the generating functional of the 4D theory according to eq.(1.1). Finally, by varying twice with respect to the boundary sources, one obtains the holographic result for $< J_\mu(x)J_\mu(y) >$.

Using Fourier-transformed gauge fields, written as $\tilde{A}_\mu(p,z) = \tilde{A}_\mu(p)\mathcal{V}(p,z)$, where $\tilde{A}_\mu(p,z)$ and $\tilde{A}_\mu(p)$ are the Fourier transforms of $A_\mu(x,z)$ and the source $A_\mu(x)$ respectively, one gets:

$$S^{(2)}_{\mathrm{AdS}_5} = -\int \frac{d^4p}{(2\pi)^4}\tilde{A}^\mu(p)\tilde{A}_\mu(p)\left(w(z)\partial_z\mathcal{V}(p,z)\right)|_{z=\epsilon}. \tag{2.3}$$

The bulk-to-boundary propagator, $\mathcal{V}(p,z)$, satisfies the linearized 5D equations of motion with 4D momentum, p:

$$\partial_z \left(w(z)\partial_z \mathcal{V}(p,z)\right) + p^2 w(z)\mathcal{V}(p,z) = 0 \tag{2.4}$$

and the boundary conditions

$$\mathcal{V}(p,0) = 1, \;\; \partial_z \mathcal{V}(p,z_0) = 0 \tag{2.5}$$

Varying (2.3) with respect to the boundary sources gives the scalar part of the two-point function:

$$\Sigma(p^2) = -\left(w(z)\partial_z \mathcal{V}(p,z)\right)_{z=\epsilon\to 0} \tag{2.6}$$

which is in general defined by

$$\int d^4x \; e^{ip\cdot x}\langle J_\mu(x)J_\nu(0)\rangle = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Sigma(p^2) \; . \tag{2.7}$$

Let the functions $\varphi_n(z)$ form the orthogonal basis of normalizable eigenfunctions satisfying

$$\partial_z \left(w(z)\partial_z \varphi_n(z)\right) + m_n^2 w(z)\varphi_n(z) = 0 \tag{2.8}$$

and the boundary conditions:

$$\varphi_n(0) = 0, \;\; \partial_z \varphi_n(z_0) = 0. \tag{2.9}$$

They are normalized according to the scalar product

$$< \varphi_m, \varphi_n > \equiv \int_0^{z_0} \varphi_m(z)\varphi_n(z)w(z)dz \tag{2.10}$$

in order to get from the 5D action a canonically normalized 4D kinetic term for the Kaluza-Klein modes $A_\mu^{(n)}(x,z) = A_\mu^{(n)}(x)\varphi_n(z)$. They also satisfy the completeness relation

$$\sum_{n=1}^{\infty} w(z')\varphi_n(z)\varphi_n(z') = \delta(z-z') \tag{2.11}$$

The two-point function $\Sigma(p^2)$ admits a bound-state expansion given by

$$\Sigma(p^2) = \sum_{n=1}^{\infty} \frac{F_n^2}{p^2 - m_n^2} \tag{2.12}$$

with

$$F_n^2 = \left(\left(w(z)\partial_z \varphi_n(z)\right)_{z=\epsilon\to 0}\right)^2 . \tag{2.13}$$

Eq. (2.12) shows the pole structure of the two-point function, corresponding to an infinite sum over resonances which correspond vector mesons with increasing masses m_n and decay constants F_n.

It is also useful to introduce the Green function $\mathcal{G}(p,z,z')$, which is the solution of the equation

$$\partial_z \left(w(z)\partial_z \mathcal{G}(p,z,z')\right) + p^2 w(z)\mathcal{G}(p,z,z') = \delta(z-z') \tag{2.14}$$

with the boundary conditions

$$\mathcal{G}(p,0,z') = 0, \quad \partial_z \mathcal{G}(p,z_0,z') = 0 \tag{2.15}$$

Then $\mathcal{G}(p,z,z')$ can be expanded as

$$\mathcal{G}(p,z,z') = \sum_{n=1}^{\infty} \frac{\varphi_n(z)\varphi_n(z')}{p^2 - m_n^2} \tag{2.16}$$

and the bulk-to-boundary propagator $\mathcal{V}(p,z)$ and the two-point function $\Sigma(p^2)$ can be respectively written as:

$$\mathcal{V}(p,z) = (w(z')\partial_{z'}\mathcal{G}(p,z,z'))_{z'=\epsilon\to 0} \tag{2.17}$$

and

$$\Sigma(p^2) = (w(z)\partial_z w(z')\partial_{z'}\mathcal{G}(p,z,z'))_{z=z'=\epsilon\to 0} \tag{2.18}$$

3 First order corrections due to a metric perturbation

3.1 Correction to the two-point function

Let us now suppose the conformal factor in the metric (2.1) to be written as

$$w(z) = w_0(z)(1 + h(z)), \tag{3.1}$$

with $|h(z)| < 1$, that can treated perturbatively, and $h(0) = 0$, so as not to alter the behavior of the metric on the UV boundary $z = 0$. Then, writing the bulk-to-boundary as $\mathcal{V}(p,z) = \mathcal{V}_0(p,z) + \mathcal{V}_1(p,z)$, one has that the unperturbed $\mathcal{V}_0(p,z)$ obviously satisfies (2.4) with $w(z)$ replaced by $w_0(z)$ and the boundary conditions (2.5), while the first order correction $\mathcal{V}_1(p,z)$ is a solution of

$$\partial_z \left(w_0(z)\partial_z \mathcal{V}_1(p,z)\right) + p^2 w_0(z)\mathcal{V}_1(p,z) = -w_0(z) D_1 \mathcal{V}_0(p,z), \tag{3.2}$$

where the differential operator D_1 is

$$D_1 \equiv \partial_z h(z) \partial_z, \tag{3.3}$$

and satisfies homogeneous boundary conditions

$$\mathcal{V}_1(p,0) = 0, \quad \partial_z \mathcal{V}_1(p,z_0) = 0. \tag{3.4}$$

Let $f_1(p,z)$ and $f_2(p,z)$ be two independent solutions of the differential equation for $\mathcal{V}_0(p,z)$, then, imposing the boundary conditions (2.5) it is easy to get

$$\mathcal{V}_0(p,z) = \frac{f_2'(p,z_0) f_1(p,z) - f_1'(p,z_0) f_2(p,z)}{f_2'(p,z_0) f_1(p,\epsilon) - f_1'(p,z_0) f_2(p,\epsilon)}, \tag{3.5}$$

where prime denotes derivative with respect to z. As we shall see in the next Section, in the case of flat 5D slice $\mathcal{V}_0(p,z)$ is expressed in terms of trigonometric functions, while in the case of

AdS_5 slice, $w_0(z) = 1/z$ and one gets for $\mathcal{V}_0(p,z)$ the well known ratio of combinations of Bessel functions [11, 12].

Standard methods, *e.g.* the use of the Green function, lead to the following expression for $\mathcal{V}_1(p,z)$

$$\begin{aligned}\mathcal{V}_1(p,z) = &\frac{f_2(p,\epsilon)f_1(p,z) - f_1(p,\epsilon)f_2(p,z)}{f_2'(p,z_0)f_1(p,\epsilon) - f_1'(p,z_0)f_2(p,\epsilon)}\left(\mathcal{I}_1 f_2'(z_0) - \mathcal{I}_2 f_1'(z_0)\right)\\ &+\frac{1}{c}\int_\epsilon^z \left(f_2(p,z)f_1(p,\zeta) - f_1(p,z)f_2(p,\zeta)\right) g(\zeta) w_0(\zeta) d\zeta \end{aligned} \tag{3.6}$$

where

$$g(z) \equiv -h'(z)\mathcal{V}_0'(p,z) = -D_1\mathcal{V}_0(p,z), \tag{3.7}$$

and

$$\mathcal{I}_i \equiv \frac{1}{c}\int_\epsilon^{z_0} f_i(p,\zeta)g(\zeta)w_0(\zeta)d\zeta, \ \ i = 1,2 \tag{3.8}$$

and $c = w_0(z)W(z)$, where $W(z) = f_2'(p,z)f_1(p,z) - f_1'(p,z)f_2(p,z)$ is the Wronskian of the two functions $f_1(p,z)$ and $f_2(p,z)$.

Previous formulae have been written with an UV cut-off $z = \epsilon > 0$ to comply with UV divergences which my appear in explicit examples, *e.g.* AdS_5, due to the divergent behavior of the conformal factor $w_0(z)$ as $z \to 0$.

From the expressions for $\mathcal{V}_0(p,z)$ and $\mathcal{V}_1(p,z)$ one obtains the first order corrected two point function

$$\Sigma(p^2) = \Sigma_0(p^2) + \Sigma_1(p^2) = -\left(w_0(z)\partial_z\mathcal{V}_0(p,z)\right)_{z=\epsilon\to 0} - \left(w_0(z)\partial_z\mathcal{V}_1(p,z)\right)_{z=\epsilon\to 0} \tag{3.9}$$

where we have assumed that $w_0(z)h(z) \to 0$ for $z \to 0$ to get rid of an additional term.

Explicit use of the Green function method to get the first order correction to the two-point function due to metric perturbation in the AdS_5 case was made in [9], where it is also mentioned that the method had been used by the authors of [10]. The expressions in [9] are readily obtained from (3.6) with $f_1(p,z)$ and $f_2(p,z)$ being, in that case, Bessel functions. We shall deal with the AdS_5 HW model in Sec. 4.2.

Although (3.9) gives the solution to the problem of finding the first order correction to the two-point function, due to metric perturbations, it rests on the choice of two solutions $f_1(p,z)$ and $f_2(p,z)$ of eq.(2.4). It would be more satisfying if the first order correction could be written in term of the unperturbed bulk-to-boundary propagator (3.5) and of the perturbation operator D_1 (3.3). Indeed, it is the case. Using the expression (3.5) for $\mathcal{V}_0(p,z)$ and further manipulating the expression (3.9), the first order correction to the two-point function can be written in the following two equivalent forms obtained one from the other by integration by parts:

$$\Sigma_1(p^2) = - < \mathcal{V}_0(p,z), D_1\mathcal{V}_0(p,z) >_0 = \int_\epsilon^{z_0} w_0(z)h(z)\left((\mathcal{V}_0'(p,z))^2 - p^2\mathcal{V}_0(p,z)^2\right)dz \tag{3.10}$$

where $< f,g >_0$ denotes the scalar product (2.10) with respect to the unperturbed metric.

Both expressions in (3.10) have interesting interpretations. The first shows the first order correction to the two-point function as the "matrix element" of the perturbation operator (3.3)

on the unperturbed bulk-to-boundary propagator $\mathcal{V}_0(p,z)$, with the scalar product defined by the unperturbed metric. The second expression in (3.10) shows that it is given by the 5D action on-shell, but with the perturbed conformal factor of the metric $w_1(z) = w_0(z)h(z)$.

The last result suggests the existence of a simpler derivation. In fact, as we shall see in a moment, it is a direct consequence of the AdS/CFT recipe (1.1) of identifying the 4D effective action with the 5D action evaluated on-shell.

In momentum space, the z-dependent part of the 5D bulk action on-shell is given by

$$S_{M_5}[\mathcal{V}] = -\int_0^{z_0} w(z)\left((\mathcal{V}'(p,z))^2 - p^2\mathcal{V}(p,z)^2\right)dz, \tag{3.11}$$

where $\mathcal{V}(p,z)$ is the bulk-to-boundary propagator of 5D vectors in the metric background with warp factor $w(z)$. If we write $w(z) = w_0(z)(1+h(z))$ and look for a perturbative solution of the form $\mathcal{V}(p,z) = \mathcal{V}_0(p,z) + \mathcal{V}_1(p,z)$ we have at first order:

$$\begin{aligned} S_{M_5}[\mathcal{V}] = S^{(0)}_{M_5}[\mathcal{V}_0] + &< \left[\frac{\delta S^{(0)}_{M_5}}{\delta\mathcal{V}}\right]_{\mathcal{V}_0}, \mathcal{V}_1(p,z) >_0 \\ &- \int_0^{z_0} w_0(z)h(z)\left((\mathcal{V}_0'(p,z))^2 - p^2\mathcal{V}_0(p,z)^2\right)dz, \end{aligned} \tag{3.12}$$

On-shell, the first term becomes a boundary term, i.e. the 0-th order term in (3.9), while the second term, proportional to the unperturbed equation of motion, vanishes and we are left with just the last one, which is precisely what we got in (3.10).

As we have already mentioned, one can use (3.10), for instance, to evaluate sub-leading corrections to the asymptotic expansion of $\Sigma(Q^2)$ for large Euclidean momentum $Q^2 = -p^2$. More interestingly, one can use it in order to extract corrections to the parameters, masses and decay constants, appearing in the bound-state decomposition of the two-point function, by studying the structure of pole singularities of the two-point function. In fact, the correction alters both the position of the poles and the corresponding residues. There are, however, some little subtleties in this procedure which we shall illustrate in Sec.4.

3.2 Corrections to the resonance wave functions

The two-point function can be written as the bound-state decomposition (2.12), where masses and decay constants are related to the mass eigenvalues and to the values, weighted by the metric conformal factor, of the derivatives of the wave functions $\varphi_n(z)$ at the origin. In presence of a deformation of the metric, wave functions are modified too. First order corrections can be evaluated using perturbation theory for eigenfunction. The derivation is straightforward, but leads to final formulae slightly different from those familiar from quantum mechanics. The reason is that the deformation of the metric does not only affect the wave-function equations, but also the scalar product entering in the normalization condition (2.10). This leads to a new diagonal term in the first order correction of $\varphi_n(z)$, which is normally absent in perturbation theory.

If $\lambda_n^{(0)} \equiv -(m_n^{(0)})^2$ and $\varphi_n^{(0)}(z)$ denote the eigenvalues and the eigenfunctions of the unperturbed case $w(z) = w_0(z)$, then the first order corrections $\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)}$ and $\varphi_n(z) = \varphi_n^{(0)}(z) + \varphi_n^{(1)}(z)$

are given by the following expressions

$$\lambda_n^{(1)} = <\varphi_n^{(0)}, D_1\,\varphi_n^{(0)}>_0 \tag{3.13}$$

$$\begin{aligned}\varphi_n^{(1)}(z) &= -\frac{1}{2} <\varphi_n^{(0)}, \varphi_n^{(0)} h>_0 \varphi_n^{(0)}(z) \\ &+ \sum_{m\neq n} \frac{<\varphi_m^{(0)}, D_1\,\varphi_n^{(0)}>_0}{\lambda_m^{(0)} - \lambda_n^{(0)}} \varphi_m^{(0)}(z),\end{aligned} \tag{3.14}$$

where D_1 is defined in (3.3). The first term in (3.14) is the one due to the fact that the deformation of the metric affects also the scalar product. In explicit examples, it is numerically relevant.

From (3.13) and (3.14) one gets the the corrections to masses and coupling constants:

$$m_n = m_n^{(0)} \left(1 - \frac{1}{2} \frac{<\varphi_n^{(0)}, D_1\,\varphi_n^{(0)}>_0}{(m_n^{(0)})^2}\right) \tag{3.15}$$

$$F_n = F_n^{(0)} \left(1 + \frac{(\varphi_n^{(1)})'(z)}{(\varphi_n^{(0)})'(z)}\right)_{z=\epsilon\to 0} \tag{3.16}$$

It may seem not evident that the corrections to masses and decay constants obtained here generate, through the bound-state decomposition (2.12), the same correction to the two-point (3.10) deduced in the previous Section, but this is the case, as explicitly shown in the Appendix.

4 Perturbing the metric of 5D Hard-Wall models

4.1 5D flat Hard Wall model

The case of gauge vector field in a flat extra-dimension slice $0 < z < z_0$, *i.e.* with warp factor $w(z) = 1$, has a nice interpretation as the continuum limit of deconstruction models [13]. Its simplicity allows for analytic calculations of the first order perturbative corrections to masses and coupling constants due to a metric perturbation. We report here the corresponding formulae. Moreover, eq.(3.10) can be used to extract the sub-leading terms in the expansion of the two-point function for large Euclidean momenta $p^2 = -Q^2$ which is also an important effect of the metric deformation establishing an holographic link between metric deformations of the 5D theory and non-vanishing condensates of the 4D dual theory. In fact, in the 4D gauge gauge theory, sub-leading terms of the two-point function in the deep Euclidean kinematical region are produced by non-perturbative effects of condensates [14].

Turning to the case of flat 5D extra-dimension on a finite interval, the normalized wave-functions are

$$\varphi_n^{(0)}(z) = \sqrt{\frac{2}{z_0}} \sin\left(\kappa_{0,n}\frac{z}{z_0}\right), \quad \text{with} \quad \kappa_{0,n} = \frac{2n+1}{2}\pi, \tag{4.1}$$

with eigenvalues

$$\lambda_n^{(0)} = -(m_n^{(0)})^2 = -\left(\frac{\kappa_{0,n}}{z_0}\right)^2. \tag{4.2}$$

The bulk-to-boundary propagator is the solution of (2.4), with $w_0(z) = 1$ and satisfying the boundary conditions (2.5):

$$\mathcal{V}_0(z,p) = \left(\frac{\sin(pz_0)}{\cos(pz_0)} \sin(pz) + \cos(pz) \right) \tag{4.3}$$

Using (2.6), the two-point function can be written

$$\Sigma_0(p^2) = -\mathcal{V}_0'(0,p) = -p\tan(pz_0). \tag{4.4}$$

The bulk-to-boundary can also be expanded in series of eigenfunctions as follows

$$\mathcal{V}_0(z,p) = -\frac{2}{z_0^2} \sum_{n=0}^{\infty} \frac{\kappa_{0,n}}{p^2 - \left(\dfrac{\kappa_{0,n}}{z_0}\right)^2} \sin\left(\kappa_{0,n}\frac{z}{z_0}\right). \tag{4.5}$$

Then, the two point function admits the bound-states decomposition

$$\Sigma_0(p^2) = \frac{2}{z_0^3} \sum_{n=0}^{\infty} \frac{\kappa_{0,n}^2}{p^2 - \left(\dfrac{\kappa_{0,n}}{z_0}\right)^2}. \tag{4.6}$$

and one can read the values of the unperturbed masses of the vector resonances from the poles and their coupling constants from the corresponding residues at the pole:

$$m_n^{(0)} = \frac{\kappa_{0,n}}{z_0}, \quad F_n^{(0)} = \sqrt{\frac{2}{z_0^3}}\,\kappa_{0,n}. \tag{4.7}$$

We now use the results of Secs. 3, to evaluate the the correction to the two-point function and the effects on resonance masses and decay constants, due to a deformation of the flat metric of the form

$$h(z) = \eta\,(z/z_0)^4. \tag{4.8}$$

As we shall see in the in the next section, this choice of the deformation is suggested by the fact that, in the AdS/CFT correspondence, it would reproduce the effects of a 4D operator of conformal dimension four such as the gluon condensate. We obviously assume in our perturbative approach the dimensionless constant η to be small, and consider only corrections of first order in η. Notice that $h(0) = 0$.

Inserting $\mathcal{V}_0(z,p)$ above into the expression (3.10), the resulting integrals can be easily evaluated and one gets for the first order correction of the two-point function

$$\Sigma_1(p^2) = -\frac{\eta}{4z_0^4 p^3} \left(\frac{2pz_0(2p^2z_0^2 - 3)}{\cos^2(pz_0)} + 6\tan(pz_0) \right) \tag{4.9}$$

It is convenient to use the identity $a + \eta\, b \sim a/(1 - \eta\, b/a)$, valid to first order in η, and rewrite 1st order corrected two-point function in the form

$$\Sigma_0(p^2) + \Sigma_1(p^2) \sim -\frac{p\sin(pz_0)}{\cos(pz_0) - \eta\dfrac{1}{4z_0^4p^4}\left(\dfrac{2pz_0(2p^2z_0^2-3)}{\sin(pz_0)} + 6\cos(pz_0)\right)} \tag{4.10}$$

Perturbed masses and decay constants can be extracted from (4.10) by looking for the zeroes of the denominator, *i.e.* the positions of the poles, and then extracting the corresponding residues.

Explicitly, once that the two-point function has been put in the form of a ratio of two functions $\Sigma(p^2) = \mathcal{N}(p)/\mathcal{D}(p)$ as above, residues are evaluated by expanding, around the poles, *i.e.* around the zeros m_n of the denominator $\mathcal{D}(q)$, and thus

$$F_n^2 = 2m_n \frac{\mathcal{N}(m_n)}{\mathcal{D}'(m_n)}. \tag{4.11}$$

In a perturbative approach, $\mathcal{D}(p) = \mathcal{D}_0(p) + \eta\,\mathcal{D}_1(p)$ and, to first order in η, the following expressions for masses and residues are obtained

$$m_n = m_n^{(0)} + \eta\, m_n^{(1)} = m_n^{(0)} \left(1 - \eta \frac{\mathcal{D}_1(m_n^{(0)})}{\mathcal{D}_0'(m_n^{(0)})\, m_n^{(0)}}\right) \tag{4.12}$$

and

$$F_n = F_n^{(0)} \left(1 + \frac{\eta}{2}\left(\frac{m_n^{(1)}}{m_n^{(0)}} + \frac{\mathcal{N}_0'(m_n^{(0)})}{\mathcal{N}_0(m_n^{(0)})} m_n^{(1)} - \left(\frac{\mathcal{D}_0'(m_n^{(0)})\mathcal{D}_1'(m_n^{(0)}) - \mathcal{D}_0''(m_n^{(0)})\mathcal{D}_1(m_n^{(0)})}{\mathcal{D}_0'(m_n^{(0)})^2}\right)\right)\right) \tag{4.13}$$

In the case of 5D flat slice, previous formulae boil down to:

$$m_n = m_n^{(0)} \left(1 - \frac{\eta}{2} \frac{2\kappa_{0,n}^2 - 3}{\kappa_{0,n}^4}\right) \tag{4.14}$$

$$F_n = F_n^{(0)} \left(1 - \frac{\eta}{2} \frac{1}{\kappa_{0,n}^2}\right) \tag{4.15}$$

Corrections to masses and decay constants can also be obtained using the perturbation theory for the bound-state eigenfunctions in Sec.3. It is not difficult to check that the corrections to the masses coincide with those obtained from perturbation formula (3.13) for the eigenvalues. We have also checked that the corrections to the decay constants coincides numerically with the ones obtained using perturbation formulae (3.14) and taking first derivatives of the perturbed eigenfunctions evaluated at $z = 0$ as in (3.16).

Finally, using (4.9) with Euclidean momentum $p^2 = -Q^2$ one has in the large-Q^2 that the two-point function receives a subleading contribution from the metric deformation given by

$$\Sigma(-Q^2) \approx Q \left(1 + \frac{3\eta}{2} \frac{1}{z_0^4 Q^4}\right). \tag{4.16}$$

4.2 AdS$_5$ Hard-Wall model

As explained in the Introduction, the case of gauge vector field in a 5D AdS slice $0 < z < z_0$, i.e. with warp factor $w(z) = 1/z$, is the one originally proposed as holographic dual of QCD. There, the AdS/CFT dictionary, relating 5D fields and 4D operators is supposed to hold and it gives a

good understanding of what 4D effect one is actually evaluating. Referring to the original papers for the details, we limit ourself to repeat the same sort of calculations we did in the previous section for the 5D flat space, and concentrate again on the effects of a perturbation of the form (4.8), whose link to 4D condensates can be given a much solid ground in the AdS case [10].

Formulae become somewhat more complicated due to the appearance of Bessel functions both in the expression of the unperturbed bulk-to-boundary propagator and in the eigenfunctions. First order correction is obtained in terms of integrals containing Bessel functions, which, in principle, could be written in terms of higher transcendental functions, but for practical purposes, can be evaluated numerically, once the value of dimensional parameter z_0, giving the size of the 5D slice and the 4D energy scale at which 4D conformal invariance is broken, is fixed.

In the AdS case the eigenfunction of the n-th bound-state is given by

$$\varphi_n^{(0)}(z) = \frac{\sqrt{2}}{|J_1(\gamma_{0,n})|} \frac{z}{z_0} J_1(\gamma_{0,n}\frac{z}{z_0}), \tag{4.17}$$

with $\gamma_{0,n}$ being the n-th zero of the Bessel function of order zero, *i.e.* $J_0(\gamma_{0,n}) = 0$.

The eigenvalues are

$$\lambda_n^{(0)} = -(m_n^{(0)})^2 = -\left(\frac{\gamma_{0,n}}{z_0}\right)^2. \tag{4.18}$$

The bulk-to-boundary propagator is the solution of (2.4), with $w_0(z) = 1/z$ and satisfying the boundary conditions (2.5) is:

$$\mathcal{V}_0(p,z) = \frac{z}{\epsilon} \frac{Y_0(pz_0)J_1(pz) - J_0(pz_0)Y_1(pz)}{Y_0(pz_0)J_1(p\epsilon) - J_0(pz_0)Y_1(p\epsilon)}, \tag{4.19}$$

There is a divergence as the UV cut-off ϵ is sent to zero. A singular behavior is shown by the two-point function:

$$\Sigma_0(p^2) \approx p^2 \log\left(\frac{p\epsilon}{2}\right) - \frac{\pi p^2}{2} \frac{Y_0(pz_0)}{J_0(pz_0)}. \tag{4.20}$$

The logarithmic behavior in the UV is actually welcome since it is identified with the contribution of the parton loop at high momenta. The finite part in (4.20) contains the information on the resonance poles, which, are located at the zeroes of $J_0(pz_0)$. From the residues one gets the decay constants:

$$F_n^{(0)2} = \frac{\pi \gamma_{0,n}^3}{z_0^4} \frac{Y_0(\gamma_{0,n})}{J_1(\gamma_{0,n})}. \tag{4.21}$$

We can explicitly show the bound-state expansion of the two-point function by means of the Kneser-Sommerfeld formula [17] (valid for $z \leq z_0$)

$$\frac{Y_0(pz_0)J_0(pz) - J_0(pz_0)Y_0(pz)}{J_0(pz_0)} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{J_0(\gamma_{0,n}z/z_0)}{[J_1(\gamma_{0,n})]^2(p^2 z_0^2 - \gamma_{0,n}^2)},$$

The limit $z \to 0$ gives a logarithmically divergent series, which however correctly describes the behavior around the poles:

$$\Sigma_0(p^2) = \frac{2p^2}{z_0^2} \sum_{n=1}^{\infty} \frac{[J_1(\gamma_{0,n})]^{-2}}{p^2 - \left(\frac{\gamma_{0,n}}{z_0}\right)^2}, \tag{4.22}$$

From this expression one reads the values of the (unperturbed) masses of the vector resonances from the poles and their coupling constants from the corresponding residues at the pole. The agreement with the value (4.21), is provided by the identity $\gamma_{0,n} J_1(\gamma_{0,n}) Y_0(\gamma_{0,n}) = 2/\pi$.

We now use the results of Sec.3, to calculate the effects of the perturbation of a metric of the form (4.8) on the two-point function, and on resonance masses and decay constants. This choice of the deformation is suggested by the fact that, in the AdS/CFT correspondence, it would reproduce the effects of a 4D operator of conformal dimension four such as the gluon condensate. We again assume in our perturbative approach the dimensionless constant η to be a small, and consider only first order corrections in η.

Inserting $\mathcal{V}_0(z,p)$ of eq.(4.19) into the expression (3.10) one obtains integrals containing Bessel functions and, taking the limit $\epsilon \to 0$, the first order correction of the two-point function can be written

$$\Sigma_1(p^2) = \eta \frac{\pi^2}{p^2 z_0^4} \left[\frac{Y_0(pz_0)^2}{J_0(pz_0)^2} \mathcal{I}_{J_1 J_0}(pz_0) - \frac{Y_0(pz_0)}{J_0(pz_0)} (\mathcal{I}_{J_1 Y_0}(pz_0) + \mathcal{I}_{J_0 Y_1}(pz_0)) + \mathcal{I}_{Y_0 Y_1}(pz_0) \right], \quad (4.23)$$

where we defined the following integral of Bessel functions $Z_n(x)$:

$$\mathcal{I}_{Z_m Z_n}(x) \equiv \int_0^x Z_m(y)\, Z_n(y)\, y^4\, dy. \quad (4.24)$$

Proceeding as in the flat case, we can extract perturbed masses and decay constants from poles and residues of the perturbed two-point function. The final results are

$$m_n = m_n^{(0)} + \eta\, m_n^{(1)} = m_n^{(0)} \left(1 - \eta \frac{4 \mathcal{I}_{J_1 J_0}(\gamma_{0,n})}{\gamma_{0,n}^6 J_1(\gamma_{0,n})^2} \right) \quad (4.25)$$

$$F_n^2 = F_n^{(0)^2} \left[1 + \eta \left(-\frac{\pi}{\gamma_{0,n}^4} (\mathcal{I}_{J_0 Y_1}(\gamma_{0,n}) + \mathcal{I}_{Y_0 J_1}(\gamma_{0,n})) - \frac{1}{2} \frac{m_n^{(1)}}{m_n^{(0)}} \right.\right. \quad (4.26)$$
$$\left.\left. + \frac{\pi}{\gamma_{0,n}^4} \frac{Y_1(\gamma_{0,n})}{J_1(\gamma_{0,n})} \mathcal{I}_{J_0 J_1}(\gamma_{0,n}) + \frac{1}{4} \frac{m_n^{(1)}}{m_n^{(0)}} \gamma_{0,n} \frac{J_2(\gamma_{0,n})}{J_1(\gamma_{0,n})} - \frac{1}{2} \frac{m_n^{(1)}}{m_n^{(0)}} \gamma_{0,n} \frac{Y_1(\gamma_{0,n})}{Y_0(\gamma_{0,n})} \right) \right]$$

As an independent check, we can apply the perturbation theory of Sec.3.2 to the AdS bound-states (4.17). Using eq.(3.15) one recovers the same expression (4.25) for mass corrections; through the perturbation series (3.14) and from eq.(3.16), one finds numerical agreement with the values of the corrections to the decay constants one gets from (4.26).

In Table.1, these corrections are shown for the first few vector and axial vector resonances. In order to get these numerical values, we had to restore in our formulae the 5D coupling constant g_5^2 and fix it to $12\pi^2/N_c$ by matching the logarithmic term of the two-point function to the value of the QCD parton loop. We have also posed $z_0 = 3.1 \times 10^{-3}\,\mathrm{MeV}^{-1}$ in order to have that the mass of the first vector resonance coincide with the value of the ρ meson mass of about 776 MeV.

The values for the axial-vectors have been obtained working in the HW model of [4], where they are 5D gauge fields satisfying the same 5D field equations as the vectors, but different IR

Resonance	$m_n^{(0)}$ (MeV)	$(\delta m_n/m_n^{(0)})/\eta$	$10^{-3}\ F_n^{(0)}$ (MeV^2)	$(\delta F_n/F_n^{(0)})/\eta$
ρ^1	776	-0.15	109	-0.23
ρ^2	1781	-0.041	380	-0.044
ρ^3	2792	-0.017	747	-0.018
a_1^1	1236	0.045	223	0.049
a_1^2	2263	0.013	548	0.011
a_1^3	3282	0.006	955	0.009

Table 1: Corrections to vector and axial vector masses and decay constants due to the deformation $h(z) = \eta\,(z/z_0)^4$ of the metric of a 5D AdS slice.

boundary conditions at z_0, *i.e.* $\varphi_n^A(z_0) = 0$, in order to break chiral symmetry. This leads to different unperturbed masses, decay constant, wave-function and bulk-to-boundary propagator; however our perturbative treatment of the correction due to the metric deformation (4.8) can be done analogously to what we have explicitly illustrated in the case of vectors.

The mass corrections of the first vector and axial vector resonance, *i.e.* the ρ and the A_1, agree with those reported in Ref.[10] once our parameter η is written as in terms of their o_4, both related to the gluon condensate as follows:

$$\eta = \frac{9\pi^2}{4}\ o_4 = \frac{3\pi}{16N_c} z_0^4\,\langle \alpha_s G_{\mu\nu} G^{\mu\nu}\rangle. \tag{4.27}$$

We also notice that η is almost equal to the parameter g of Ref.[15], which expresses the gluon condensate effects in the Extended Nambu-Jona Lasinio model of low energy hadron phenomenology: one has $\eta = (9/8)(M_Q z_0)^4 g$, where M_Q is the constituent quark mass of that model.

The relation between η and the gluon condensate can be established by the fact that in presence of the metric perturbation a non-vanishing sub-leading inverse power correction appears in the two-point function at large Euclidean momentum. Explicitly in this limit and using $p^2 = -Q^2$ in (4.23) the two-point function receives a sub-leading contribution from the metric deformation given by

$$\Sigma(-Q^2) \approx -\frac{N_c}{24\pi^2} Q^2 \left[\log\left(\frac{Q\epsilon}{2}\right)^2 - \frac{16\,\eta}{3 z_0^4 Q^4}\right]. \tag{4.28}$$

The coefficient of the sub-leading term can then be matched with the one produced in QCD by a non-vanishing gluon condensate:

$$\Sigma\left(-Q^2\right) = -\frac{N_c}{24\pi^2} Q^2 \left[\log\left(\frac{Q^2}{\mu^2}\right) - \frac{\left(\pi\alpha_s\,\langle G_{\mu\nu}G^{\mu\nu}\rangle\,/N_c\right)}{Q^4} + \mathcal{O}\left(\frac{1}{Q^6}\right)\right], \tag{4.29}$$

From (4.28) and (4.29) it follows that η has the same sign of the gluon condensate, which experimentally results to be positive (see, for instance, the up-to-date discussion in [16] and references therein). The use of the value, $\langle \alpha_s G_{\mu\nu} G^{\mu\nu}\rangle \sim 6.8 \cdot 10^{-2}$ GeV^4, consistent with the present determination (see [7]) for the gluon condensate, leads to $\eta \sim 1.26$ in eq.(4.27); however the proper

value to use for holographic models should only come from a fit of the full holographic predictions to the data but, anticipating the phenomenological discussion in the next Section, we can say that it is close to a reasonable value.

It should be noticed that, in the HW model of [4], where chiral symmetry breaking is imposed through different boundary conditions for vector and axial vector gauge fields, the deformation of the metric produces the same sub-leading term at large Euclidean momentum both for the vector and the axial-vector two-point functions. The effects of different IR b.c. is indeed exponentially suppressed at large Euclidean momentum. This is in agreement with the non-chiral nature of the gluon condensate in QCD.

Let us come to a consistency check of our results. In any HW model, completeness property of the resonance wave-functions give rise to sum rules involving vector and axial vector meson masses and decay constant and other parameters of the low energy chiral lagrangian. Explicitly, one has the following sum rules [4],[18]

$$\sum_n g_n^2 = 8\,L_1 \tag{4.30}$$

$$\sum_n f_n g_n = 2L_9 \tag{4.31}$$

$$\sum_n g_n^2 m_n^2 = \frac{f_\pi^2}{3}, \tag{4.32}$$

$$\sum_n f_n g_n m_n^2 = f_\pi^2, \tag{4.33}$$

where L_1 and L_9 are two of the Gasser-Leutwyler coefficients of the $O(p^4)$ terms of the chiral Lagrangian [1], $f_\pi = 92.3$ MeV is the pion decay constant, $f_n = F_n/m_n^2$ and g_n are the pion decay constants and two-pion-one-vector couplings as they are usually written in the Chiral Lagrangian formulation of spin-1 vector field sector,

$$\mathcal{L}_V = -\frac{f_V}{2\sqrt{2}}\langle\, V_{\mu\nu}\, f_+^{\mu\nu}\,\rangle - \frac{ig_V}{2\sqrt{2}}\,\langle V_{\mu\nu}\,[\,u^\mu\,,\,u^\nu\,]\rangle + \ldots \tag{4.34}$$

where $V_{\mu\nu}$ is the V_μ field strength, $u_\mu = i\,u^\dagger\,D_\mu\,U\,u^\dagger$, with $U = u^2$ being the chiral field and $f_+^{\mu\nu} = u\,F_L^{\mu\nu}\,u^\dagger + u^\dagger\,F_R^{\mu\nu}\,u$, with $F_{L,R}^{\mu\nu}$ external sources. For the lowest resonance, the ρ, one would have $f_V \equiv f_\rho \equiv f_1$ and $g_V \equiv g_\rho \equiv g_1$. Our notations in Eq.(4.34) agree with those of Ref.[4].

Notice that in writing the sum rules (4.30-4.33) we are assuming χSB and the pion wave function of in the 5D holographic model of QCD of Ref.[4]. A different approach is followed in Ref.[3].

The physical 4D meaning of the sum rules above is that they ensure soft high energy behavior of some quantities relevant for physical processes such as the pion elastic scattering amplitude

[1]The HW models of [4] automatically enforce, in the $SU_L(3)\times SU_R(3)$ case, the relations $L_3 = 6\,L_1 = 3\,L_2$ and $L_9 + L_{10} = 0$

(4.30), the vector form factor (4.31,4.33). The sum rule (4.32) is the analog of the KSFR for the infinite tower of resonances one obtains in these HW model. Note that in models with vector meson dominance, corresponding to just the first resonance, the natural value KSFR is a factor '2' instead of '3' which is what one gets instead in these HW models.

In the expression of the L_i there enters the pion wave function $\alpha(z)$ which, for a metric of the form (2.1) is solution of the equation

$$\partial_z \left(w(z) \partial_z \alpha(z) \right) = 0, \tag{4.35}$$

with boundary conditions

$$\alpha(0) = 1, \quad \alpha(z_0) = 0. \tag{4.36}$$

In presence of a deformation of the original AdS metric, the function $\alpha(z)$ will also get corrections. At first order in η one has

$$\alpha(z) = 1 - \frac{z^2}{z_0^2} - \frac{\eta}{3} \left(\frac{z^2}{z_0^2} - \frac{z^6}{z_0^6} \right) \tag{4.37}$$

The value of g_n is obtained as a 5D overlapping integral containing $\alpha(z)$ and the vector wave-function $\varphi_n(z)$, *i.e*:

$$g_n = \frac{\sqrt{2}}{2 g_5} \int_0^{z_0} w(z) (1 - \alpha(z)^2) \varphi_n(z) dz. \tag{4.38}$$

As we may apply the perturbation formulae (3.14) to get the correction to the vector wave functions $\phi_n(z)$, we are able to evaluate numerically the corrections to the couplings g_n's. Their values for the firs few resonances are shown in the table and the plot of Fig. 1.

Resonance	$g_n^{(0)}$	$(\delta g_n / g_n^{(0)})/\eta$
ρ^1	0.0538	0.34
ρ^2	-0.0019	-2.31
ρ^3	0.0003	-3.72

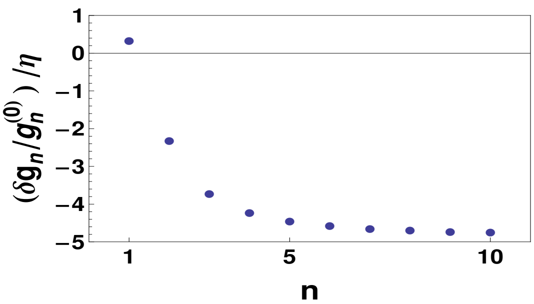

Figure 1: Corrections to the one-vector-two-pions couplings for the first few vector resonances due to the deformation $h(z) = \eta \, (z/z_0)^4$ of the metric of a 5D AdS slice.

We now report the expressions of the Gasser-Leutwyler coefficients of the chiral lagrangian obtained in the AdS HW model [4], at first order in the metric perturbation of eq.(4.8). Using $1/g_5^2 = N_c/12\pi^2$, one has:

$$
\begin{aligned}
f_\pi &= \sqrt{\frac{1}{g_5^2}\int_0^{z_0} w(z)(\alpha'(z))^2 dz} = \frac{\sqrt{N_c}}{\sqrt{6}\pi z_0}\left(1+\frac{1}{6}\eta\right) && (4.39)\\
L_1 &= \frac{1}{32 g_5^2}\int_0^{z_0} w(z)(1-\alpha(z)^2)^2 dz = \frac{11 N_c}{9216\pi^2}\left(1+\frac{2}{3}\eta\right) && (4.40)\\
L_9 &= \frac{1}{4 g_5^2}\int_0^{z_0} w(z)(1-\alpha(z)^2) dz = \frac{N_c}{64\pi^2}\left(1+\frac{25}{54}\eta\right) && (4.41)\\
L_2 &= 2L_1,\ L_3 = -6L_1,\ L_{10} = -L_9. && (4.42)
\end{aligned}
$$

Notice that the smallness of the numerical coefficients in front of η in (4.39-4.41) makes these corrections perturbative even for $\eta \sim O(1)$. When the perturbation parameter η is written in terms of o_4, following (4.27), our expressions (4.39-4.41) coincide with the ones in Ref.[10].

An important result of holographic models is that the sum rules (4.30-4.33) still hold with the perturbed values of the L_i's, of f_π and of the single resonances masses m_n, decay constants f_n and coupling to two pions g_n. As such, we can also address an issue similar to the one raised in Ref.[19] in the context of deconstruction models with unperturbed metrics. The authors wondered how fast is the convergence of the sum over the resonances in the first and second Weinberg sum rule (WSR): these sums, compared to eqs. (4.30-4.33) involve also axial vectors. Their findings was that actually a large number of terms was needed; for instance in the case of deconstruction with "cosh z" background, the one leading to AdS in the continuum limit, and twenty lattice sites, practically all resonances had to be considered in order to get a small deviation in the WSR.

Authors of Ref. [20] address a different question in QCD deconstruction: how many sites are needed to have a a good phenomenological picture. They find that already a model with three or four sites reproduces the relevant features of holographic models of QCD.

These two issues led us to investigate how good is the convergence of the sum rules in a perturbed metrics in Eqs. (4.30-4.33), i.e. how many resonances are needed to achieve a good descriptions of Eqs. (4.39-4.41). This is even more reasonable, if some pattern of low resonance dominance shows up in the theory. Indeed, for all sum rules the convergence remains quite good even after the corrected values of m_n, f_n and g_n are inserted, and what is more important, the numerical values converge rapidly with just few terms in the sum to those obtained from the perturbations of f_π, L_1 and L_9, what gives us an important check. Actually, a dominance of the few lowest resonances appear, as it is illustrated in Fig. 2, where saturation of each sum rule by the lowest resonances is shown in both the unperturbed and in the first order corrected case.

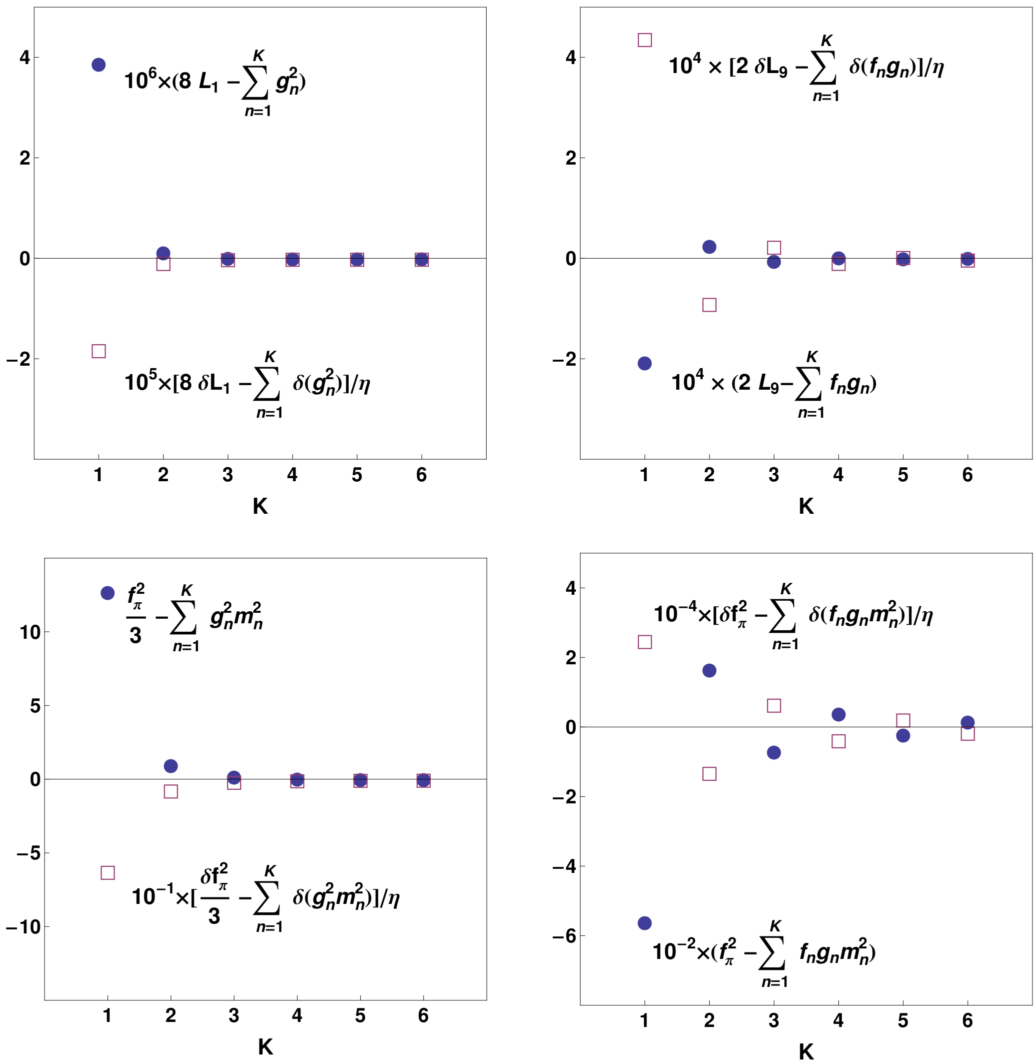

Figure 2: The four plots show the result of adding the first K terms in the two sum rules (4.30-4.33) in the HW model of Ref. [4], both in the case of pure AdS metric (dots) and in presence of first order corrections in the metric deformation (4.8) (squares). To put both in the same plot, the differences in the unperturbed and in the perturbed case have been magnified by suitable factors, as indicated. One easily recognizes that the sum rules are dominated by the contribution of the first few resonances also in presence of the metric perturbation. In the plots, the perturbative corrections in eqs. (4.39),(4.40) and (4.41) and for m_n, f_n and g_n (re-scaled by η) have been denoted by δf_π, δL_1 and δL_9 etc..

5 Conclusions

Having evaluated in eqs.(4.39-4.41), the gluonic contributions to the L_i's and to f_π it is interesting to check if they lead to a better agreement with the known experimental values. Since we have been able to evaluate the effects of the gluonic condensate on masses, decay constants and vector-to-two-pion couplings of single vector resonances we can add also these parameter to the discussion.

Before embarking in the numerical analysis, we should stress however that the model we are

considering has intrinsic approximations that cannot be underestimated when trying to compare it with the real dynamics of low energy QCD. As in any Large-N_c model of QCD we are neglecting order $\mathcal{O}(1/N_c)$ effects, such as $\pi\pi$-loops. We also know that the model does not reproduce the Regge behavior. The same mechanism of chiral symmetry breaking is strongly model dependent. We have followed the one proposed in Ref.[4], where the pion is introduced as a 5D *non-local* object linked to a Wilson line between the two boundary of the AdS_5-slice. This leads to the expression (4.37) for the pion wave function. In Refs. [3], although the description of vector resonances is the same, the spontaneous chiral symmetry breaking is triggered by a 5D scalar field, dual to the $\bar{q}q$ operator and not by IR boundary conditions. We shall try to take care of these caveats in the numerical discussion that follows. As in Ref.[4] we work in the chiral limit.

We want to distinguish between the predictions of the model for those observables which involve the spectrum of resonance as a whole as in the case for the Gasser-Leutwyler coefficients and the pion decay constant and the predictions for parameters related to a single resonance. Moreover, motivated by the previous discussion, we shall estimate the systematic error of the model for the prediction of the pion decay constant and of single resonance parameters to be of about $10-20\%$. The numerical analysis which we have performed gives indication in favor of this approach.

We refer to Table 2 and Figure 3 to illustrate the results of the analysis: five Gasser-Leutwyler coefficients, the pion decay constant and the mass, the decay constant and the coupling to two pions of the lowest vector resonance, identified with the ρ meson, are the observables that we are using in different fits.

As benchmarks, in the first column of Table 2 are shown the values that one obtains in the absence of gluon condensate, $\eta = 0$, with $N_c = 3$, *i.e.* correct matching in the UV of the vector two-point function with the QCD parton loop, and with IR scale of the model z_0 fixed by using the physical value $m_\rho = 776$ MeV as input. The values of the L_i's appear to be underestimated.

As an attempt to improve the predictions of the model, in absence of gluon condensate, one could abandon the UV matching and let N_c vary. This was done in Ref. [4], where N_c was fixed by using the value of f_π as input, with z_0 still fixed by the physical value of m_ρ. There resulted an improved agreement with experimental values, at the cost of having $N_c = 4.3$, signalling that important physical effects were still not captured by the holographic model without the condensate.

Let us now discuss the inclusion of the gluon condensate. As we explained before, we shall focus on the prediction of the model for observables which involve the complete spectrum of resonance as is the case for the Gasser-Leutwyler coefficients and the pion decay constant. In the second column of Table 2, we report the values obtained by the fit of the L_i and f_π, using the expressions (4.39-4.42), with an estimated theoretical error on f_π of 20%. The values of η and z_0 which minimize the χ^2 are $\eta = 0.73$ and $z_0 = 2.7 \times 10^{-3}$ MeV^{-1} (not far from the value 3.1×10^{-3} MeV^{-1} which would be obtained by fitting m_ρ without condensate), with χ^2/d.o.f.=3.2/4. The values obtained for η and z_0 have been used to compute the values of m_ρ, f_ρ and g_ρ. The 1σ region for (z_0, η) is light shaded in the first contour plot of Fig.3.

The third column of Table 2 shows the values obtained by adding also the values of the physical parameters of the lowest vector resonance, the ρ, with an estimated theoretical error of 10%. The values of η and z_0 which minimize the χ^2 are $\eta = 0.89$ and $z_0 = 2.7 \times 10^{-3}$ MeV^{-1}, with χ^2/d.o.f.=8.0/ 7. The 1σ region for (z_0, η) is light shaded in the second contour plot of Fig.3. The determination of the best-χ^2 values is sharpened at the cost of having an higher χ^2 per d.o.f.

	Vanishing gluon condensate $\eta = 0,\ z_0(m_\rho^{phys})$	Fit of L_i and $f_\pi(20\%)$ $(\chi^2/\text{d.o.f.}) = (3.2/4)$	Fit of $L_i,\ f_\pi(20\%)$ and $(m_\rho,\ f_\rho,\ g_\rho)(10\%)$ $(\chi^2/\text{d.o.f.}) = (8.0/7)$	Experiment
$10^3 L_1$	0.36	0.54	0.58	0.4 ± 0.3
$10^3 L_2$	0.72	1.08	1.16	1.35 ± 0.3
$10^3 L_3$	-2.2	-3.2	-3.5	-3.5 ± 1.1
$10^3 L_9$	4.7	6.3	6.7	6.9 ± 0.7
$10^3 L_{10}$	-4.7	-6.3	-6.7	-5.5 ± 0.7
f_π (MeV)	72.8	92.4	95.6	92.3 ± 0.3
m_ρ (MeV)	$776^\sharp$	$783^\dagger$	769	775.8 ± 0.5
f_ρ	0.18	$0.19^\dagger$	0.19	$0.20 \pm 0.02^*$
g_ρ	0.054	$0.067^\dagger$	0.070	$0.087 \pm 0.009^*$

Table 2: Theoretical predictions and experimental values for the five Gasser-Leutwyler coefficients $L_1,\ L_2, L_3, L_9, L_{10}$, the pion decay constant f_π and the mass m_ρ, the decay constant f_ρ and the coupling g_ρ of ρ to two pions. The first column refers to the values obtained in the absence of gluon condensate, $\eta = 0$, with $N_c = 3$, and with IR scale of the model z_0 fixed by using the physical value $m_\rho = 776$ MeV (the $^\sharp$ denotes that there m_ρ is an input parameter). The second and third column show the result of fitting the experimental values of the L_i, with their experimental values, together with f_π (third column) and together also with m_ρ, f_ρ and g_ρ (The values in the second column have a † to remember that they have been evaluated after the fit of the other observable had been done). As explained in the text, to f_π and to the physical quantities of the ρ meson have been assigned theoretical errors of 20% and 10% respectively. The * on the values of f_ρ and g_ρ in the last column indicate that errors have been estimated in the Chiral Lagrangian theoretical framework.

Using Eq.(4.27) one can derive from $\eta = 0.89$ and $z_0 = 2.7 \times 10^{-3}$ MeV^{-1} a prediction for the condensate $\langle \alpha_s G_{\mu\nu} G^{\mu\nu} \rangle = (8.7\,^{+9.4}_{-4.2}) \times 10^{-2}\,\text{GeV}^4$ which is compatible with the average experimental value $\langle \alpha_s G_{\mu\nu} G^{\mu\nu} \rangle_{\text{exp.}} = (6.8 \pm 1.13) \times 10^{-2}\,\text{GeV}^4$ reported in Ref.[16].

We believe that this simple analysis already shows that gluon condensate corrections evaluated through metric deformation in HW models leads to results qualitatively in agreement to what we expect from QCD. We notice also that the trends of the numerical values of the Gasser-Leutwyler coefficients, that we have found, in the presence of gluon condensate, and also those of vector and axial vector masses and decay constants agree with those evaluated in the ENJL model in [15].

Now we want to observe another hint towards the connection between the Chiral Quark Model [6, 7] and the AdS Hard-Wall model. As remarked in Ref. [21], it just happens in the Chiral Quark Model that, with neglect of the gluonic couplings, for some of the $\mathcal{O}(p^4)$ terms in the QCD effective action $(L_1,\ L_2,\ L_3,\ L_9,\ L_{10})$ one gets finite results in the UV and IR domains, *i.e.* cut-off independent values; of course this does not hold once the gluonic corrections are turned on. The same is happening for the same L_i's in the AdS HW model: no dependence on z_0. Here we want to stress that this gives further support to the AdS picture vs the flat metric where the same L_i's show an explicit dependence on the IR cut-off, z_0.

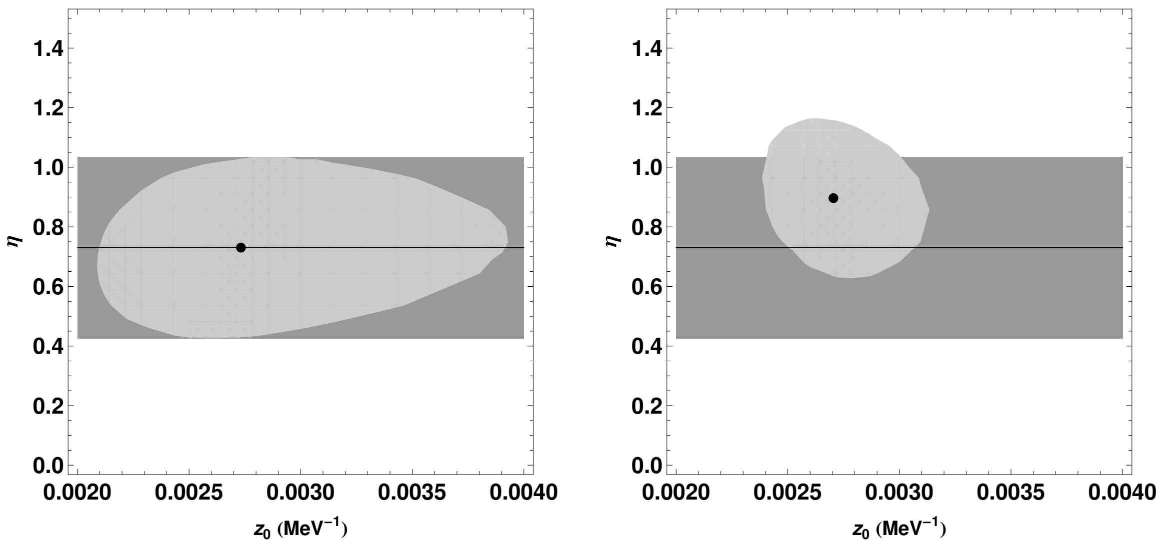

Figure 3: Central values and 1σ regions for the values of the IR scale z_0 and the condensate parameter η are represented by a black dot and a light shaded region respectively for the two fits in Table 2. The left plot, with central values $\eta = 0.73$ and $z_0 = 2.7 \times 10^{-3}$ MeV^{-1}, corresponds to the second column in Table 2: fit of the L_i's and f_π. The right plot, with central values $\eta = 0.89$ and $z_0 = 2.7 \times 10^{-3}$ MeV^{-1}, corresponds to third column in Table 2: fit of the L_i's, f_π, m_ρ, f_ρ and g_ρ. The dark shaded band in both plots is the 1σ region for η obtained by a fit of L_i's values only, with the central value $\eta = 0.73$ shown by the horizontal line.

In this paper we have considered the holographic description of the effects of a gluon condensate in terms of a suitable deformation of the metric of a 5D Hard Wall model. We have not considered the effects of the $\langle \bar{q}q \rangle$ condensate since our focus was on low energy QCD physics, once χSB has been already triggered by some mechanism. Since the estimated value for the gluon condensate is phenomenologically acceptable, our treatment seems appropriate. Moreover, the quark condensate should not be considered in the comparison with the Chiral Quark model.

Comparing with previous literature, we have computed for the first time the corrections to the resonance decay constants, f_n and vector-to two-pion couplings, g_n and we have shown the validity of the sum rules even in presence of gluon condensate: this has been the core of our calculation. We have also discussed the case of flat metric. We have found substantial evidence toward an analogy of the AdS HW holographic model and the Chiral Quark model. We have also performed a novel numerical analysis, which leads to a satisfactory value for the gluon condensate, showing that the model is able to capture the relevant physics properties.

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A Appendix

In this Appendix we give a proof of the equivalence of the two perturbative methods developed in Sec.3. We neglect the issue of the convergence of the bound-state expansion (2.12).

Let us first show the useful identity

$$\sum_n \sum_m \frac{w_0(0)^2 \varphi_n^{(0)'}(0) \varphi_m^{(0)'}(0)}{p^2 - (m_n^{(0)})^2} < \varphi_n^{(0)}, \varphi_m^{(0)} h >_0 = 0 \tag{A.1}$$

The LHS of the previous expression can be written

$$\sum_n \frac{w_0(0)^2 \varphi_n^{(0)'}(0)}{p^2 - (m_n^{(0)})^2} < \varphi_n^{(0)}, \sum_m \varphi_m^{(0)'}(0) \varphi_m^{(0)} h >_0 \tag{A.2}$$

Then,

$$\begin{aligned} &< \varphi_n^{(0)}, \sum_m \varphi_m^{(0)'}(0) \varphi_m^{(0)} h >_0 = \int_0^{z_0} \varphi_n^{(0)}(z) \sum_m w_0(z) \varphi_m^{(0)'}(0) \varphi_m^{(0)}(z) h(z) dz = \\ &\frac{d}{dz'} \left(\int_0^{z_0} \varphi_n^{(0)}(z) \sum_m w_0(z) \varphi_m^{(0)}(z') \varphi_m^{(0)}(z) h(z) dz \right)_{z'=0} = \frac{d}{dz'} \left(\varphi_m^{(0)}(z') h(z') \right)_{z'=0} = 0 \end{aligned} \tag{A.3}$$

where we used the completeness relation (2.11) for the unperturbed eigenfunctions, and the last equality is a consequence of the boundary condition (2.9) and of the vanishing of $h(z)$ at the origin.

Using the identity (A.1) one can show, for instance, that the two expressions which are obtained inserting the bound-state expansion for $\mathcal{V}(p, z)$ in the two different ways of expressing the first order correction to the two-point function in (3.10), actually coincide and give

$$- < \mathcal{V}_0, D_1 \mathcal{V}_0 >_0 = - \sum_n \sum_m \frac{w_0(0)^2 \varphi_n^{(0)'}(0) \varphi_m^{(0)'}(0)}{(p^2 - (m_n^{(0)})^2)(p^2 - (m_m^{(0)})^2)} < \varphi_n^{(0)}, D_1 \varphi_m^{(0)} >_0 . \tag{A.4}$$

This expression has to be identical to the one which we obtain inserting in the bound-state expansion (2.12) the perturbative corrections (3.13) and (3.14) to masses and eigenfunctions, *i.e.*

$$\Sigma(p^2) = \sum_n \frac{(w(0) \varphi_n'(0))^2}{p^2 - m_n^2} = \sum_n \frac{(w_0(0)(1 + h(0))(\varphi_n^{(0)'}(0) + \varphi_n^{(1)'}(0)))^2}{p^2 - ((m_n^{(0)})^2 + (m_n^{(1)})^2)} \tag{A.5}$$

Using (3.13) and (3.14), one has

$$\frac{1}{p^2 - ((m_n^{(0)})^2 + (m_n^{(1)})^2)} = \frac{1}{p^2 - (m_n^{(0)})^2} \left(1 + \frac{(m_n^{(1)})^2}{p^2 - (m_n^{(0)})^2} \right) = \frac{1}{p^2 - (m_n^{(0)})^2} \left(1 - \frac{< \varphi_n^{(0)}, D_1 \varphi_n^{(0)} >_0}{p^2 - (m_n^{(0)})^2} \right) \tag{A.6}$$

and then

$$\Sigma(p^2) = \Sigma_0(p^2) - \sum_n \frac{(w_0(0)\varphi_n^{(0)'}(0))^2}{(p^2-(m_n^{(0)})^2)^2} < \varphi_n^{(0)}, D_1\, \varphi_n^{(0)} >_0 +$$

$$2\sum_n \frac{w_0(0)^2\varphi_n^{(0)'}(0)}{p^2-(m_n^{(0)})^2}\left(-\frac{1}{2} < \varphi_n^{(0)}, \varphi_n^{(0)} h >_0 \varphi_n^{(0)'}(0) - \sum_{m\neq n} \frac{< \varphi_m^{(0)}, D_1\varphi_n^{(0)} >_0}{(m_n^{(0)})^2-(m_m^{(0)})^2}\varphi_m^{(0)'}(0)\right) \tag{A.7}$$

The next step is to notice that, as

$$\frac{1}{p^2-(m_n^{(0)})^2}\,\frac{1}{p^2-(m_m^{(0)})^2} = \frac{1}{(m_n^{(0)})^2-(m_m^{(0)})^2}\left(\frac{1}{p^2-(m_n^{(0)})^2} - \frac{1}{p^2-(m_m^{(0)})^2}\right), \tag{A.8}$$

it is convenient to express the matrix elements $< \varphi_m^{(0)}, D_1\varphi_n^{(0)} >_0$ as the sum of the symmetric and antisymmetric parts, which we denote by $S_{mn} = S_{nm}$ and $A_{mn} = -A_{nm}$ respectively. With a little effort, one can show that

$$S_{mn} = \frac{1}{2}\left((m_n^{(0)})^2 + (m_m^{(0)})^2\right) < \varphi_m^{(0)}, \varphi_n^{(0)} h >_0 \; - \; < \varphi_m^{(0)'}, \varphi_n^{(0)'} h >_0,$$

$$A_{mn} = \frac{1}{2}\left((m_n^{(0)})^2 - (m_m^{(0)})^2\right) < \varphi_m^{(0)}, \varphi_n^{(0)} h >_0 \;.$$

Then, the first order correction to $\Sigma(p^2)$ in (A.7) splits into four terms as follows:

$$\begin{aligned} &-\sum_n \frac{(w_0(0)\varphi_n^{(0)'}(0))^2}{(p^2-(m_n^{(0)})^2)^2} < \varphi_n^{(0)}, D_1\varphi_n^{(0)} >_0 - \sum_n \frac{(w_0(0)^2\varphi_n^{(0)'}(0))^2}{p^2-(m_n^{(0)})^2} < \varphi_m^{(0)}, \varphi_n^{(0)} h >_0 \\ &-2\sum_n\sum_{m\neq n} \frac{w_0(0)^2\varphi_m^{(0)'}(0)\varphi_n^{(0)'}(0)}{(p^2-(m_n^{(0)})^2)((m_n^{(0)})^2-(m_n^{(0)})^2)} S_{mn} \\ &-\sum_n\sum_{m\neq n} \frac{w_0(0)^2\varphi_m^{(0)'}(0)\varphi_n^{(0)'}(0)}{p^2-(m_n^{(0)})^2} < \varphi_m^{(0)}, \varphi_n^{(0)} h >_0 \end{aligned} \tag{A.9}$$

Using the identity (A.1), it can be shown that the sum of the second and the fourth term in (A.9) is zero. Using the symmetry of S_{mn} and the identity (A.8), the third term in (A.9) can be rewritten

$$-\sum_n\sum_{m\neq n} \frac{w_0(0)^2\varphi_n^{(0)'}(0)\varphi_m^{(0)'}(0)}{(p^2-(m_n^{(0)})^2)(p^2-(m_m^{(0)})^2)} < \varphi_n^{(0)}, D_1\varphi_m^{(0)} >_0, \tag{A.10}$$

The first term in (A.9) provides the diagonal terms not present in (A.10) and allows to fully recover, as promised, the expression (A.4).

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